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Stress-Based Elastodynamic Discrete Laminated Plate Theory

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A static laminated plate theory based on an assumed piecewise linear through-the-thickness in-plane stress distribution has been extended to include inertia effects. Based on this in-plane stress distribution assumption, out-of-plane shear and normal stress component distributions were derived from the three-dimensional equations of motion, resulting in six nonzero stress components. Hamilton's variational principle was used to derive the plate equations of motion, the plate constitutive relationships, and the interface continuity equations. The governing equations were written in a form that is self-adjoint with respect to the convolution bilinear mapping. The resulting system of equations for a single lamina consists of 25 field equations in terms of 9 weighted displacement field variables, 10 stress and moment resultant field variables, and 6 out-of-plane shear and normal stress boundary field variables. For the laminated system, the mixed formulation enforces both traction and displacement continuity at lamina interfaces as it satisfies layer equilibrium. A finite element formulation based on a specialized form of the governing functional was developed. The method is illustrated with results of a free vibration analysis of sandwich and homogeneous plates for which exact solutions are available.

Introduction

CURRENT dynamic laminated plate theories that assume a through-the-thickness displacement distribution are capable of predicting global response (e.g., lower mode frequencies and deflections) satisfactorily but not local stress behavior. When studying the behavior of laminates subjected to transient loads that often produce high stress gradients, an accurate description of the stress field is imperative. To model localized behavior, several discrete laminate theories that analyze laminates on a layerwise basis have been developed. One such theory for statics by Pagano[1] has been shown to accurately model stress fields in regions of high stress gradients for free-edge delamination coupons. This theory, which makes no explicit displacement distribution assumption, has six nonzero stress components and enforces traction and displacement continuity at lamina interfaces. In this paper, Pagano's theory is extended to include inertia effects. Based on an assumed linear through-the-thickness in-plane stress distribution, derivations for the out-of-plane shear and normal stresses using the three-dimensional equations of motion are presented.

Literature Review

The laminated plate theories used to analyze laminated plates can be categorized as smeared or discrete. Smeared theories combine the properties of individual lamina to obtain representative laminate properties to describe the plate, whereas discrete theories analyze laminates on a layerwise basis. A thorough review of the state-of-the-art laminated plate theories can be found in Ref. 2.

Plate theories approximate the three-dimensional elasticity equations by assuming a distribution of either stress or displacement through the thickness of the plate. In general, displacement-based theories satisfy kinematic conditions but not equilibrium, whereas stress-based theories satisfy equilibrium but not kinematic relations. The use of either a stress- or a displacement-based approach should depend on the required critical output. Displacement-based theories generally predict displacements with the greater degree of accuracy, whereas stress-based theories will in general give more accurate stress predictions.

Reissner[3] presented a stress-based plate theory for the transverse bending of homogeneous plates that assumes a linear variation of the in-plane stress components and uses the equilibrium equations to solve for the remaining out-of-plane stress components. Applications of this theory, which include the effect of transverse shearing strain and transverse normal stress, demonstrated the importance of the contribution of the out-of-plane shear strain energy to the total strain energy for bending of plates. Pagano[1] extended Reissner's work to laminated plates by assuming in-plane stresses to be linear through the thickness of each lamina while enforcing both continuity of stresses and displacements at lamina interfaces. Pagano presented results closely approximating free-edge stress fields in composite laminate delamination coupons.[4]

A dynamic stress-based approach using Reissner's[3] out-of-plane shear and normal stress distribution to analyze homogeneous isotropic plates was presented by Voyiadjis and Baluch.[5] The in-plane stress components were obtained by integrating appropriate strain-displacement relationships combined with linear elastic constitutive relations. The in-plane stress components were expressed as functions of shear stress resultants and moment resultants as well as midplane displacements. Expressions for the six stress components were subsequently substituted into the equations of motion. The plate equations of motion and a governing differential equation in terms of the transverse midplane displacement and the in-plane moment resultants were derived. A comparison of wave speeds obtained by Voyiadjis and Baluch with exact solutions shows that the two compare closely over a wide spectrum of wavelengths.

Development of the Present Dynamic Theory

Since Pagano's static stress-based discrete laminated plate theory overcomes many of the difficulties associated with other plate theories for predicting localized stress behavior, the theory was generalized to include inertia effects. For an assumed through-the-thickness distribution of the in-plane stress components, the out-of-plane shear and normal stresses are derived from the equations of equilibrium yielding six nonzero stress components. Additionally, layer equilibrium, interface traction, and displacement continuity are satisfied. The present theory is a generalization of the

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theory of Ref. 1 to include inertia effects. Details of the derivation that are not presented here can be found in Ref. 2.

Derivation of Lamina Stresses

In the following, a rectangular plate of uniform thickness h is assumed to be homogeneous and linearly elastic. The coordinate system used in the derivation of the equations is shown in Fig. 1. The origin is located at the midplane of the plate that is bounded by $x_1 = \pm a, x_2 = \pm b$, and $x_3 = \pm h/2$. The differential equations of motion for three-dimensional elasticity are written as

$$\sigma_{ij,j} + f_i - \dot{p}_i = 0 \tag{1}$$

where σ_{ij} are the components of the symmetric Cauchy stress tensor, f_i are the components of the body force vector per unit volume, and p_i are the momentum densities of the medium (mass density times the velocity vector). The superposed dots represent differentiation with respect to time. Standard index notation is used in which Latin indices take on the range of values from 1 to 3 and Greek indices take on the values of 1 and 2. Summation on repeated indices is implicit, and subscripts following a comma denote differentiation with respect to the spatial coordinates indicated by the subscripts. The mass density is assumed to be independent of time; therefore

$$\dot{p}_i = \rho \ddot{u}_i \tag{2}$$

This assumption has little consequence in the confines of the linearly elastic small deformation theory used herein. Considering a monoclinic linear elastic material, the constitutive equations are

$$\varepsilon_{\alpha\beta} = S_{\alpha\beta\gamma\delta}\sigma_{\gamma\delta} + S_{\alpha\beta 33}\sigma_{33} \tag{3}$$

$$\varepsilon_{\alpha 3} = \text{™}_{3\alpha} = 2S_{\alpha 3\beta 3}\sigma_{\beta 3} \tag{4}$$

$$\varepsilon_{33} = S_{33\gamma\delta}\sigma_{\gamma\delta} + S_{3333}\sigma_{33} \tag{5}$$

where S_{ijkl} are rate-independent components of the elastic compliance tensor. For small deformations, the kinematic relations for linear elasticity are

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) = u_{(i,j)} \tag{6}$$

where ε_{ij} is the symmetric strain tensor. Integrating Eq. (6) yields the strain-displacement relationships

$$u_1(x_1, x_2, x_3, t) = u_1(-a, x_2, x_3, t) + \int_{-a}^{x_1} \varepsilon_{11} d\eta_1 \tag{7}$$

$$u_2(x_1, x_2, x_3, t) = u_2(x_1, -b, x_3, t) + \int_{-b}^{x_2} \varepsilon_{22} d\eta_2 \tag{8}$$

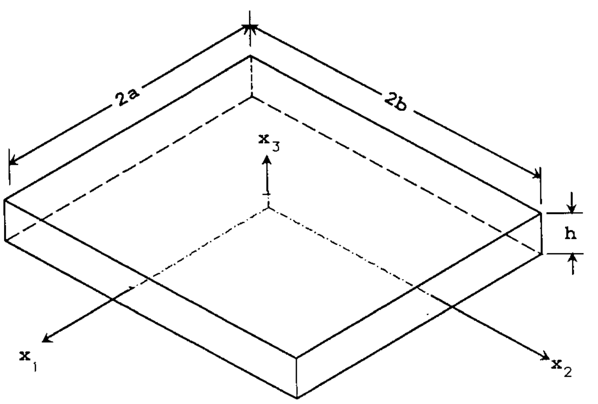


Fig. 1 Plate geometry and coordinate system.

$$u_3(x_1, x_2, x_3, t) = u_3\left(x_1, x_2, -\frac{h}{2}, t\right) + \int_{-h/2}^{x_3} \varepsilon_{33} d\eta_3 \tag{9}$$

where, for example, $u_1(-a, x_2, x_3, t)$ are the values of the in-plane displacements in the x_1 direction at the boundary $-a$ for x_2, x_3, and t. Substituting relationships (3–5) into Eqs. (7–9), we obtain displacements expressed in terms of stresses and boundary-displacement terms. For example, Eq. (9) becomes

$$u_3(x_1, x_2, x_3, t) = u_3\left(x_1, x_2, -\frac{h}{2}, t\right) + \int_{-h/2}^{x_3} (S_{33\gamma\delta}\sigma_{\gamma\delta} + S_{3333}\sigma_{33})\, d\eta_3 \tag{10}$$

The plate equations of motion are developed by substituting Eqs. (7–9), Eq. (2), and stress-strain relationships (3–5) into Eq. (1). The equations of motion are then written as

$$\sigma_{\alpha\beta,\beta} + \sigma_{\alpha 3,3} + f_\alpha - \rho\ddot{u}_\alpha = 0 \tag{11}$$

$$\sigma_{\alpha 3,\alpha} + \sigma_{33,3} + f_3 - \rho\ddot{u}_3 = 0 \tag{12}$$

where

$$\ddot{u}_1(x_1, x_2, x_3, t) = \ddot{u}_1(-a, x_2, x_3, t) + \int_{-a}^{x_1} (S_{11\gamma\delta}\ddot{\sigma}_{\gamma\delta} + S_{1133}\ddot{\sigma}_{33})\, d\eta_1 \tag{13}$$

$$\ddot{u}_2(x_1, x_2, x_3, t) = \ddot{u}_2(x_1, -b, x_3, t) + \int_{-b}^{x_2} (S_{22\gamma\delta}\ddot{\sigma}_{\gamma\delta} + S_{2233}\ddot{\sigma}_{33})\, d\eta_2 \tag{14}$$

$$\ddot{u}_3(x_1, x_2, x_3, t) = \ddot{u}_3\left(x_1, x_2, -\frac{h}{2}, t\right) + \int_{-h/2}^{x_3} (S_{33\gamma\delta}\ddot{\sigma}_{\gamma\delta} + S_{3333}\ddot{\sigma}_{33})\, d\eta_3 \tag{15}$$

Adopting a notation similar to that introduced by Pagano,[1] let

$$(\tilde{u}, \bar{u}, \hat{u}) \equiv \int_{-h/2}^{h/2} \left(1, \frac{2x_3}{h}, \frac{4x_3^2}{h^2}\right) \frac{2}{h} u \, dx_3 \tag{16}$$

To reduce the equations to expressions in two dimensions, the equations of motion are integrated over the transverse dimension x_3. The mass density ρ is taken to be constant over the thickness of the plate. Integrating Eqs. (11) and (12), the first moment of Eq. (11) over the thickness of the plate gives the plate equations of motion as

$$N_{\alpha\beta,\beta} + (\overset{+}{\sigma}_{\alpha 3} - \overset{-}{\sigma}_{\alpha 3}) + F_\alpha - \frac{\rho h}{2}\ddot{\tilde{u}}_\alpha = 0 \tag{17}$$

$$V_{\alpha,\alpha} + (\overset{+}{\sigma}_{33} - \overset{-}{\sigma}_{33}) + F_3 - \frac{\rho h}{2}\ddot{\tilde{u}}_3 = 0 \tag{18}$$

$$M_{\alpha\beta,\beta} + \frac{h}{2}(\overset{+}{\sigma}_{\alpha 3} + \overset{-}{\sigma}_{\alpha 3}) - V_\alpha - \frac{\rho h^2}{4}\ddot{\bar{u}}_\alpha = 0 \tag{19}$$

where the following definitions for weighted accelerations are introduced:

$$\frac{h}{2}\tilde{\ddot{u}}_1 \equiv \int_{-h/2}^{h/2} \ddot{u}_1(-a, x_2, x_3, t)\,dx_3 + \int_{-a}^{x_1} (S_{11\gamma\delta}\ddot{N}_{\gamma\delta} + S_{1133}\ddot{N}_{33})\,d\eta_1 \tag{20}$$

$$\frac{h}{2}\tilde{\ddot{u}}_2 \equiv \int_{-h/2}^{h/2} \ddot{u}_2(x_1, -b, x_3, t)\,dx_3 + \int_{-b}^{x_2} (S_{22\gamma\delta}\ddot{N}_{\gamma\delta} + S_{2233}\ddot{N}_{33})\,d\eta_2 \tag{21}$$

$$\frac{h}{2}\tilde{\ddot{u}}_3 \equiv h\ddot{u}_3\left(x_1, x_2, -\frac{h}{2}, t\right) + \int_{-h/2}^{h/2}\int_{-h/2}^{x_3} (S_{33\gamma\delta}\ddot{\sigma}_{\gamma\delta} + S_{3333}\ddot{\sigma}_{33})\,d\eta_3\,dx_3 \tag{22}$$

$$\frac{h^2}{4}\bar{\ddot{u}}_1 \equiv \int_{-h/2}^{h/2} \ddot{u}_1(-a, x_2, x_3, t)\,x_3\,dx_3 + \int_{-a}^{x_1} (S_{11\gamma\delta}\ddot{M}_{\gamma\delta} + S_{1133}\ddot{M}_{33})\,d\eta_1 \tag{23}$$

$$\frac{h^2}{4}\bar{\ddot{u}}_2 \equiv \int_{-h/2}^{h/2} \ddot{u}_2(x_1, -b, x_3, t)\,x_3\,dx_3 + \int_{-b}^{x_2} (S_{22\gamma\delta}\ddot{M}_{\gamma\delta} + S_{2233}\ddot{M}_{33})\,d\eta_2 \tag{24}$$

The in-plane force and moment resultants for $\sigma_{\alpha\beta}$ are given by $N_{\alpha\beta}$ and $M_{\alpha\beta}$, respectively. The in-plane force resultant for $\sigma_{\alpha 3}$ is given by V_α, and the out-of-plane generalized force resultants for σ_{33} are expressed as

$$(N_{33}, M_{33}) \equiv \int_{-h/2}^{h/2} (1, x_3)\,\sigma_{33}\,dx_3 \tag{25}$$

The boundary terms for the top and bottom surfaces of the plate are defined as

$$\overset{+}{\sigma}_{i3} \equiv \sigma_{i3}\left(x_1, x_2, \frac{h}{2}, t\right) \tag{26}$$

$$\overset{-}{\sigma}_{i3} \equiv \sigma_{i3}\left(x_1, x_2, \frac{-h}{2}, t\right) \tag{27}$$

Assuming that the body force per unit volume f_i is constant through-the-thickness, define

$$F_i \equiv hf_i \tag{28}$$

Equations (17–19) are the integral form of the generalized plate equations of motion applicable to wave propagation in the plane of the plate. It is noted that the plate formulation thus far is exact; i.e., no assumptions regarding the distribution of stresses or displacements have been made.

The variables $N_{\alpha\beta}$, $M_{\alpha\beta}$, and V_α are expressed as generalized stress and moment resultants of the in-plane components of the stress tensor. The inverse relationship, i.e., the stress distribution for a given $N_{\alpha\beta}$, $M_{\alpha\beta}$, and V_α is not uniquely defined. However, if an assumption is made regarding the distribution of some components of σ_{ij}, the distribution of the others may be determined. For a homogeneous plate, Reissner[3] assumed a linear distribution of in-plane stresses $\sigma_{\alpha\beta}$ through the thickness. Substituting the linear relationship $\sigma_{\alpha\beta} = A_{\alpha\beta} + B_{\alpha\beta}x_3$ into the definitions for in-plane stress and moment resultants and solving for $A_{\alpha\beta}$ and $B_{\alpha\beta}$ in terms of the stress and moment resultants yield

$$\sigma_{\alpha\beta} = \frac{N_{\alpha\beta}}{h} + \frac{12M_{\alpha\beta}}{h^3}x_3 \tag{29}$$

Having specified the distribution of the in-plane stress components, the out-of-plane normal and shear stresses can be derived by integrating the equations of motion (1). Integrating the first two of Eqs. (1) through the thickness of the plate and rearranging give

$$\sigma_{\alpha 3} = \overset{-}{\sigma}_{\alpha 3} - \int_{-h/2}^{x_3} \sigma_{\gamma\delta,\beta}\,d\eta_3 - \int_{-h/2}^{x_3} f_\alpha\,d\eta_3 + \rho\int_{-h/2}^{x_3} \ddot{u}_\alpha\,d\eta_3 \tag{30}$$

Differentiating Eq. (29) with respect to x_β and substituting into Eq. (30) gives

$$\sigma_{\alpha 3} = \overset{-}{\sigma}_{\alpha 3} - \int_{-h/2}^{x_3} \left(\frac{N_{\alpha\beta,\beta}}{h} + \frac{12M_{\alpha\beta,\beta}}{h^3}\eta_3\right) d\eta_3 - \int_{-h/2}^{x_3} f_\alpha\,d\eta_3 + \rho\int_{-h/2}^{x_3} \ddot{u}_\alpha\,d\eta_3 \tag{31}$$

Combining Eqs. (17) and (19) with Eq. (31), performing the appropriate integration, and simplifying, results in the following expression for the out-of-plane shear stresses:

$$\sigma_{\alpha 3} = (\overset{+}{\sigma}_{\alpha 3} - \overset{-}{\sigma}_{\alpha 3})\frac{x_3}{h} + \frac{1}{4}(\overset{+}{\sigma}_{\alpha 3} + \overset{-}{\sigma}_{\alpha 3})\left(\frac{12x_3^2}{h^2} - 1\right) + \frac{3V_\alpha}{2h}\left(1 - \frac{4x_3^2}{h^2}\right) - \frac{\rho}{2}\left[\tilde{\ddot{u}}_\alpha\left(x_3 + \frac{h}{2}\right) + \bar{\ddot{u}}_\alpha\left(\frac{3x_3^2}{h} - \frac{3h}{4}\right)\right] + \rho\int_{-h/2}^{x_3} \ddot{u}_\alpha\,d\eta_3 \tag{32}$$

To derive the expression for σ_{33}, the third equation of motion is integrated through the thickness direction x_3 yielding

$$\sigma_{33} = \overset{-}{\sigma}_{33} - \int_{-h/2}^{x_3} \sigma_{\alpha 3,\alpha}\,d\eta_3 - \int_{-h/2}^{x_3} f_3\,d\eta_3 + \rho\int_{-h/2}^{x_3} \ddot{u}_3\,d\eta_3 \tag{33}$$

Differentiating Eq. (32) with respect to x_α and substituting into Eq. (33), one finds that the out-of-plane normal stress becomes

$$\sigma_{33} = \frac{1}{2}(\overset{+}{\sigma}_{33} + \overset{-}{\sigma}_{33}) - \left[(\overset{+}{\sigma}_{\alpha 3,\alpha} + \overset{+}{\alpha}_{\alpha 3,\alpha})\left(\frac{x_3^2}{2h} - \frac{h}{8}\right) \right.$$
$$\left. \times \left(\frac{x_3^3}{h^2} - \frac{x_3}{4}\right) - \frac{3}{2}(\overset{+}{\sigma}_{33} \times \overset{-}{\sigma}_{33})\left(\frac{x_3}{h} - \frac{4x_3^3}{3h^3}\right) \right]$$
$$+ \frac{\rho}{2}\left[\ddot{\overset{+}{u}}_{\alpha,\alpha}\left(\frac{x_3^2}{2} + \frac{hx_3}{2} + \frac{h^2}{8}\right) + \ddot{\overset{-}{u}}_{\alpha,\alpha}\left(\frac{x_3^3}{h} - \frac{3hx_3}{4} - \frac{h^2}{4}\right) \right.$$
$$\left. + \ddot{\overset{+}{u}}_3\left(\frac{2x_3^3}{h^2} - \frac{3x_3}{2} - \frac{h}{2}\right)\right] - \rho \int_{-h/2}^{x_3} \int_{-h/2}^{\eta_3} \ddot{u}_{\alpha,\alpha}\, d\zeta_3\, d\eta_3$$
$$+ \rho \int_{-h/2}^{x_3} \ddot{u}_3\, d\eta_3 - f_3\left(\frac{2x_3^3}{h^2} - \frac{x_3}{2}\right) \tag{34}$$

To express the out-of-plane stress components, given by Eqs. (32) and (34), in terms of generalized stress resultants and boundary stress terms, one must develop expressions for the weighted displacements (accelerations) terms. Each of the terms involving generalized displacements or integrals of displacements and their time derivatives can be expressed in terms of integrals of generalized stresses and boundary stress terms and their spatial and time derivatives. Upon developing these expressions and substituting into Eq. (34), one derives an integral-differential expression for σ_{33}:

$$\sigma_{33} = \frac{1}{4}(\overset{+}{\sigma}_{33} + \overset{-}{\sigma}_{33})\left(\frac{12x_3^2}{h^2} - 1\right) + \frac{1}{4}(\overset{+}{\sigma}_{33} - \overset{-}{\sigma}_{33})$$
$$\times \left(\frac{40x_3^3}{h^3} - \frac{6x_3}{h}\right) + \frac{3N_{33}}{2h}\left(1 - \frac{4x_3^2}{h^2}\right) + \frac{15M_{33}}{h^2}\left(\frac{2x_3}{h} - \frac{8x_3^3}{h^3}\right)$$
$$+ \frac{\rho}{h}\left\{\left[S_{\xi\xi 33}\left(\frac{h}{2}\ddot{N}_{33} - \ddot{M}_{33}\right) - S_{3333}\int_{-h/2}^{h/2}\int_{-h/2}^{x_3} \ddot{\sigma}_{33}\, d\eta_3\, dx_3\right]\right.$$
$$\left. \times \left(\frac{10x_3^3}{h^2} + \frac{3x_3^2}{h^2} - \frac{3x_3}{2} - \frac{h}{4}\right)\right\} - \frac{\rho}{h}(S_{\xi\xi 33} - S_{3333})$$
$$\times \left[\int_{-h/2}^{h/2}\int_{-h/2}^{x_3}\int_{-h/2}^{\eta_3} \ddot{\sigma}_{33}\, d\zeta_3\, d\eta_3\, dx_3\left(\frac{6x_3^2}{h^2} - \frac{3}{2}\right)\right.$$
$$+ h\int_{-h/2}^{x_3}\int_{-h/2}^{\eta_3} \ddot{\sigma}_{33}\, d\zeta_3\, d\eta_3$$
$$\left. + \int_{-h/2}^{h/2} x_3 \int_{-h/2}^{x_3}\int_{-h/2}^{\eta_3} \ddot{\sigma}_{33}\, d\zeta_3\, d\eta_3\, dx_3\left(\frac{120x_3^3}{h^4} - \frac{30x_3}{h^2}\right)\right] \tag{35}$$

The out-of-plane components of shear stress, σ_{13} and σ_{23}, can be expressed as a function of generalized stress resultants, boundary stress and displacement terms, and their spatial integrals and time derivatives as well as spatial integrals and time derivatives of σ_{33}. Once σ_{33} is expressed in terms of these variables, the out-of-plane shear stress components can likewise be expressed in terms of the generalized stress resultants and boundary stress and displacement terms. Therefore, the out-of-plane stress components are formulated based on an assumed linear distribution of in-plane stresses.

An approximate solution to the integral-differential equation, Eq. (35), for σ_{33} can be obtained by using Picard's iteration method.[6] The approximate solution is obtained by selecting the static terms of Eq. (35), i.e., terms that do not involve time derivatives, and back substituting into Eq. (35). The resulting higher order approximate solution for σ_{33}, which includes terms with second-order derivatives in time, is then substituted back into Eq. (35), yielding another approximation that includes fourth-order derivatives in time. This procedure can be followed as many times as necessary to achieve a desired degree of accuracy for σ_{33}.

Evaluating the order of h for each term in the approximation for σ_{33} provides information about the contribution of each term to the total stress σ_{33}. It is first noted that N_{33} is of order 1 in h, i.e., $\mathcal{O}(h^1)$, and M_{33} is $\mathcal{O}(h^2)$. It can be shown that the static terms in the approximation for Eq. (35) are $\mathcal{O}(h^0)$, and terms involving second-order time differentiation are $\mathcal{O}(h^2)$. Furthermore, terms involving fourth-order derivatives in time are of order $\mathcal{O}(h^4)$. Assuming that the time derivatives of the terms in the approximation remain bounded, we may select h to be as small as necessary to make the higher order terms negligible. In this work, the following approximation is used to develop the governing field equations:

$$\sigma_{33} = \frac{1}{4}(\overset{+}{\sigma}_{33} + \overset{-}{\sigma}_{33})\left(\frac{12x_3^2}{h^2} - 1\right) + \frac{1}{4}(\overset{+}{\sigma}_{33} - \overset{-}{\sigma}_{33})$$
$$\times \left(\frac{40x_3^3}{h^3} - \frac{6x_3}{h}\right) + \frac{3N_{33}}{2h}\left(1 - \frac{4x_3^2}{h^2}\right) + \frac{15M_{33}}{h^2}\left(\frac{2x_3}{h} - \frac{8x_3^3}{h^3}\right) \tag{36}$$

Equation (36) is the same expression derived by Pagano[1] for σ_{33} in his static theory. Having defined an expression for σ_{33}, one can determine the out-of-plane shear stress components:

$$\sigma_{13} = (\overset{+}{\sigma}_{13} - \overset{-}{\sigma}_{13})\frac{x_3}{h} + \frac{1}{4}(\overset{+}{\sigma}_{13} + \overset{-}{\sigma}_{13})\left(\frac{12x_3^2}{h^2} - 1\right)$$
$$+ \frac{3V_1}{2h}\left(1 - \frac{4x_3^2}{h^2}\right) + \frac{\rho}{h}\int_{-a}^{x_1} S_{1133}\left\{\left[\frac{h}{4}(\ddot{\overset{+}{\sigma}}_{33} + \ddot{\overset{-}{\sigma}}_{33})\right.\right.$$
$$\left. - \frac{1}{2}\ddot{N}_{33}\right]\left(\frac{4x_3^3}{h^3} - x_3\right) + \left[\frac{h^2}{12}(\ddot{\overset{+}{\sigma}}_{33} - \ddot{\overset{-}{\sigma}}_{33}) - \ddot{M}_{33}\right]$$
$$\left. \times \left(\frac{30x_3^4}{h^4} - \frac{9x_3^2}{h^2} + \frac{3}{8}\right)\right\} d\eta_1 - \frac{\rho}{h}\int_{-h/2}^{h/2} \ddot{u}_1(-a, x_2, x_3, t)\, dx_3$$
$$\times \left(x_3 + \frac{h}{2}\right) - \frac{\rho}{h}\int_{-h/2}^{h/2} \ddot{u}_1(-a, x_2, x_3, t)\, x_3\, dx_3\left(\frac{6x_3^2}{h^2} - \frac{3}{2}\right)$$
$$+ \rho\int_{-h/2}^{x_3} \ddot{u}_1(-a, x_2, x_3, t)\, d\eta_3 \tag{37}$$

and

$$\sigma_{23} = (\overset{+}{\sigma}_{23} - \overset{-}{\sigma}_{23})\frac{x_3}{h} + \frac{1}{4}(\overset{+}{\sigma}_{23} + \overset{-}{\sigma}_{23})\left(\frac{12x_3^2}{h^2} - 1\right)$$
$$+ \frac{3V_2}{2h}\left(1 - \frac{4x_3^2}{h^2}\right) + \frac{\rho}{h}\int_{-b}^{x_2} S_{2233}\left\{\left[\frac{h}{4}(\ddot{\overset{+}{\sigma}}_{33} + \ddot{\overset{-}{\sigma}}_{33})\right.\right.$$
$$\left. - \frac{1}{2}\ddot{N}_{33}\right]\left(\frac{4x_3^3}{h^3} - x_3\right) + \left[\frac{h^2}{12}(\ddot{\overset{+}{\sigma}}_{33} - \ddot{\overset{-}{\sigma}}_{33}) - \ddot{M}_{33}\right]$$
$$\left. \times \left(\frac{30x_3^4}{h^4} - \frac{9x_3^2}{h^2} + \frac{3}{8}\right)\right\} d\eta_2 - \frac{\rho}{h}\int_{-h/2}^{h/2} \ddot{u}_2(x_1, -b, x_3, t)\, dx_3$$
$$\times \left(x_3 + \frac{h}{2}\right) - \frac{\rho}{h}\int_{-h/2}^{h/2} \ddot{u}_2(x_1, -b, x_3, t)\, x_3\, dx_3\left(\frac{6x_3^2}{h^2} - \frac{3}{2}\right)$$
$$+ \rho\int_{-h/2}^{x_3} \ddot{u}_1(x_1, -b, x_3, t)\, d\eta_3 \tag{38}$$

In this section approximate expressions for the out-of-plane stress components satisfying the three-dimensional equations of motion have been derived for small h.

Governing Field Equations

The present initial boundary-value problem consists of determining internal stresses in an anisotropic elastic laminate subjected to prescribed tractions and/or displacements on its boundary. An energy formulation using Hamilton's principle was used to derive the governing equations. This principle states that the total energy of the body at any time greater than or equal to zero is the integral over time of the kinetic energy minus the potential energy or

$$J = \int_0^t (T - \Pi)\, dt \tag{39}$$

Substituting expressions for the kinetic energy and the complementary form of the potential energy into Eq. (39) and taking the first variation with respect to σ_{ij} and u_i, we arrive at

$$\delta J = \int_0^t \int_V \left[\left(\frac{u_{i,j} + u_{j,i}}{2} - \frac{\partial W}{\partial \sigma_{ij}}\right)\delta\sigma_{ij} - (\sigma_{ij,j} + f_i - \rho\ddot{u}_i)\delta u_i\right] dV\, dt$$
$$+ \int_0^t \int_{S_u} (u_i - s_i)\,\delta(\eta_j\sigma_{ij})\, dS\, dt - \int_0^t \int_{S_\sigma} (\eta_j\sigma_{ij} - t_i)\,\delta u_i\, dS\, dt$$
$$+ \int_0^t \int_{S'_u} (u'_i - g'_i)\,\delta(\eta_j\sigma'_{ij})\, dS\, dt$$
$$- \int_0^t \int_{S'_\sigma} (\eta_j\sigma'_{ij} - h'_i)\,\delta u_i\, dS\, dt$$
$$- \int_V (\rho\dot{u}_i\, du_i)\big|_0^t\, dV = 0 \tag{40}$$

where the volume of the body is represented by V and the surface of the body by S. The portion of the boundary where traction is prescribed is denoted by S_σ, and the portion of the boundary where displacement is prescribed is represented by S_u. The surfaces S_u and S_σ are disjoint sets of S, such that $S_u \cup S_\sigma = S$. The boundary conditions for the problem are

$$u_i = s_i \text{ on } S_u \qquad \eta_j\sigma_{ij} = t_i \text{ on } S_\sigma$$
$$u'_i = g'_i \text{ on } S'_u \qquad \eta_j\sigma'_{ij} = h'_i \text{ on } S'_\sigma \tag{41}$$

where the primes indicate internal surfaces of jump discontinuity.

The appropriate field equations are found by setting to zero the expression corresponding to each of the arbitrary admissible variations of the field variables. Terms involving fourth-order derivatives in time are assumed to be negligible, reducing the problem to one involving only second-order derivatives in time.

Ten constitutive equations, 9 equations of motion and 6 interface displacement equations constitute a set of 25 governing field equations with 25 field variables. By comparison, Pagano's static formulation[1] consisted of 23 field equations and 23 field variables. The present theory includes two additional higher order weighted displacement field variables. These two additional displacement field variables appear in the field equations as accelerations.

Self-Adjoint Formulation

To implement the present theory into a finite element formulation, a self-adjoint form of the governing equations is desirable so that a Ritz-type variational formulation can be used. The definitions of self-adjointness, bilinear mapping, and Gateaux differential for the initial boundary-value problem are discussed and defined in the procedure developed and outlined by Sandhu[7] and Sandhu and Salaam[8] and further developed by Al-Ghothani[9] for the variational formulation of linear problems. Since the field equations as derived from Hamilton's variational principle are not in a self-adjoint form, the bilinear mapping introduced by Gurtin[10,11] was used. Gurtin's procedure eliminates time derivatives from the field equations and introduces the initial conditions explicitly. The following nondegenerate convolution bilinear mapping was proposed by Gurtin:

$$B_{R\times t}(u, v) = \int_R (u * v)\, dR = \int_R dR \int_0^t u(x, t - \tau)\, v(x, \tau)\, d\tau \tag{42}$$

Throughout, an asterisk denotes the convolution integral that satisfies the distributive, associative, and commutative laws.

The equations to this point have been written for a single lamina. To account for coupling between layers in a laminate system, hereafter we write the equations in terms of specific layers. For the problem at hand, we have a laminate composed of N layers designated such that layer $k = 1$ is the top lamina and layer $k = N$ is the bottom lamina of the laminate. Note that the field equations must be satisfied in each layer of the laminate. A representative four-layer laminate system is depicted in Fig. 2.

The following field variable definitions are introduced for the kth layer of the laminate:

$$\bar{v}_\rho^{(k)} \equiv {}^1\!/_2\, \tilde{u}_\rho^{(k)} \tag{43}$$

$$\bar{\phi}_\rho^{(k)} \equiv \frac{3}{h_k}\bar{u}_\rho^{(k)} m \tag{44}$$

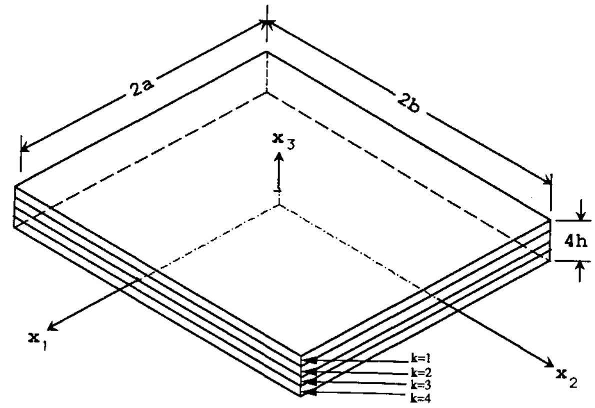


Fig. 2 Coordinate system for a four-layer laminate.

$$
[A]^{(k)} = \begin{bmatrix}
-\rho h_k & 0 & 0 & 0 & 0 & \tfrac{1}{2}t^*\Gamma_1 & 0 & 0 & 0 & 0 \\
0 & -\dfrac{\rho h_k^3}{12} & 0 & 0 & 0 & 0 & 0 & \tfrac{1}{2}t^*\Gamma_1 & 0 & -t^* \\
0 & 0 & -\dfrac{5\rho h_k}{6} & -\dfrac{\rho h_k}{8} & 0 & 0 & 0 & 0 & 0 & t^*\dfrac{\partial}{\partial\gamma} \\
0 & 0 & -\dfrac{\rho h_k}{8} & \dfrac{3\rho h_k}{32} & 0 & 0 & 0 & -\dfrac{3\rho}{20}S^{(k)}_{33\alpha\beta} & -\dfrac{3\rho}{20}S^{(k)}_{3333} & 0 \\
0 & 0 & 0 & 0 & \dfrac{3\rho h_k}{4} & -\dfrac{\rho h_k}{4}S^{(k)}_{33\alpha\beta} & -\dfrac{\rho h_k}{4}S^{(k)}_{3333} & 0 & 0 & 0 \\
-\tfrac{1}{2}t^*\Gamma_2 & 0 & 0 & 0 & -\dfrac{\rho h_k}{4}S^{(k)}_{\mu\rho 33} & \dfrac{1}{h_k}t^*S^{(k)}_{\mu\rho\alpha\beta} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\dfrac{\rho h_k}{4}S^{(k)}_{3333} & 0 & -\dfrac{1}{h_k}t^*S^{(k)}_{3333} & 0 & 0 & 0 \\
0 & -\tfrac{1}{2}t^*\Gamma_2 & 0 & -\dfrac{3\rho}{20}S^{(k)}_{\mu\rho 33} & 0 & 0 & 0 & \dfrac{12}{h_k^3}t^*S^{(k)}_{\mu\rho\alpha\beta} & 0 & 0 \\
0 & 0 & 0 & -\dfrac{3\rho}{20}S^{(k)}_{3333} & 0 & 0 & 0 & 0 & \dfrac{12}{h_k^3}t^*S^{(k)}_{3333} & 0 \\
0 & -t^*\delta_{\alpha\rho} & -t^*\dfrac{\partial}{\partial\rho} & 0 & 0 & 0 & 0 & 0 & 0 & \dfrac{24}{5h_k}t^*S^{(k)}_{\rho 3\gamma 3}
\end{bmatrix}
$$

$$\bar{v}_3^{(k)} \equiv \frac{3}{4}(\bar{u} - \hat{u}_3)^{(k)} \tag{45}$$

$$\bar{\phi}_3^{(k)} \equiv (5\hat{u}_3 - \tilde{u}_3)^{(k)} \tag{46}$$

The quantities $\overset{+}{u}_i$ and $\overset{-}{u}_i$ are defined as $u_i(x_1, x_2, h/2, t)$ and $u_i(x_1, x_2, -h/2, t)$ respectively, and the interface continuity conditions for the laminate system are

$$\sigma_{i3}^{-(k)} = \sigma_{i3}^{+(k+1)} \tag{47}$$

$$u_i^{-(k)} = u_i^{+(k+1)} \tag{48}$$

Having defined the kinematic field variables by Eqs. (43–46), one can perform a restatement of the field equations to transform them to a form that is self-adjoint with respect to the convolution bilinear mapping, Eq. (42). The self-adjoint form of the governing field equations can be expressed symbolically in operator matrix form as

$$[A]^{(k)}\{u\}^{(k)} + [B]^{(k)}\{\sigma\}^{-(k)} + [C]^{(k)}\{\sigma\}^{+(k)}$$

$$+ [D_u]^{(k)}\{F\}^{(k)} + \{E_u\}^{(k)} + \{Z_u\}^{(k)} = 0 \tag{49}$$

where $[A]^{(k)}$, $[B]^{(k)}$, $[C]^{(k)}$, and $[D_u]^{(k)}$ are operator matrices for the kth layer; $\{u\}^{(k)}$ and $(\sigma)^{\pm(k)}$ are sets of field variables; $\{F\}^{(k)}$ are generalized body force terms; $\{E_u\}^{(k)}$ are in-plane boundary terms; and $\{Z_u\}^{(k)}$ are the initial conditions. Explicitly, the self-adjoint linear operator $[A]^{(k)}$ is given by
with

$$\Gamma_1 = \left(\delta_{\alpha\gamma}\frac{\partial}{\partial\beta} + \delta_{\beta\gamma}\frac{\partial}{\partial\alpha}\right)$$

and

$$\Gamma_2 = \left(\delta_{\mu\gamma}\frac{\partial}{\partial\rho} + \delta_{\rho\gamma}\frac{\partial}{\partial\mu}\right)$$

where $\delta_{\alpha\gamma}$ is the Kronecker delta. The field variables for the problem are given by

$$\{u\}^{(k)T} = \left[\bar{v}_\gamma^{(k)}, \bar{\phi}_\gamma^{(k)}, \bar{v}_3^{(k)}, \bar{\phi}_3^{(k)}, \tilde{u}_3^{(k)}, N_{\alpha\beta}^{(k)}, N_{33}^{(k)}, M_{\alpha\beta}^{(k)}, M_{33}^{(k)}, V_\gamma^{(k)}\right]^T \tag{50}$$

$$\{\sigma\}^{\pm(k)} = \begin{bmatrix} \sigma_{\gamma 3}^{\pm(k)} \\ \sigma_{33}^{\pm(k)} \end{bmatrix} \tag{51}$$

The diagonal operators of $[A]$ are self-adjoint, and the off-diagonal operators constitute adjoint pairs with respect to the convolution bilinear mapping. The operators $[B]$, $[C]$, and $[D_u]$ along with $\{F\}$, $\{E_u\}$, and $\{Z_u\}$ are presented in the Appendix.

The interface continuity equations can also be expressed symbolically in operator matrix form as

$$[\Lambda]^{(k)}\{\sigma\}^{-(k-1)} + [\bar{B}]^{(k)}\{u\}^{(k)} + [\Xi]^{(k)}\{\sigma\}^{-(k)}$$

$$+ [\bar{C}]^{(k+1)}\{u\}^{+(k+1)} + [\bar{\Lambda}]^{(k+1)} + [D_S]^{(k)}\{F\}^{(k)}$$

$$+ \{E_S\}^{(k)} + \{Z_S\}^{(k)} = 0 \tag{52}$$

The continuity of displacements at the lamina interfaces, $u_i^{-(k)} = u_i^{+(k+1)}$, is inherent in Eq. (52) and the continuity of stresses, $\sigma_{i3}^{-(k)} = \sigma_{i3}^{+(k+1)}$, is included in the finite element mass and stiffness assembly. The operator matrices $[\Lambda]$, $[\bar{B}]$, $[\Xi]$, $[\bar{C}]$, $[\bar{\Lambda}]$, and $[D_S]$ as well as $\{F\}$, $\{E_S\}$, and $\{Z_S\}$are given in the Appendix.

Based on the set of governing field equations, a governing functional was derived.[7,8]. For finite element implementation, specializations of the governing functional were performed. By reducing the required order of differentiability of selected field variables, the admissible function space to include lower order functions or polynomials. The requirements for the admissible function space to identically satisfy the constitutive relationships allowed the dependent field variables $N_{\alpha\beta}^{(k)}$, $N_{33}^{(k)}$, $M_{\alpha\beta}^{(k)}$, $M_{33}^{(k)}$, and $V_\alpha^{(k)}$ to be

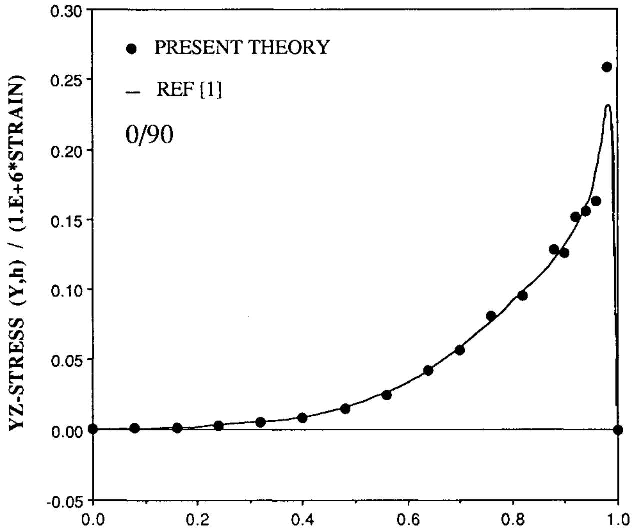


Fig. 3 Distribution of yz stress along 0/90 cross-ply interface.

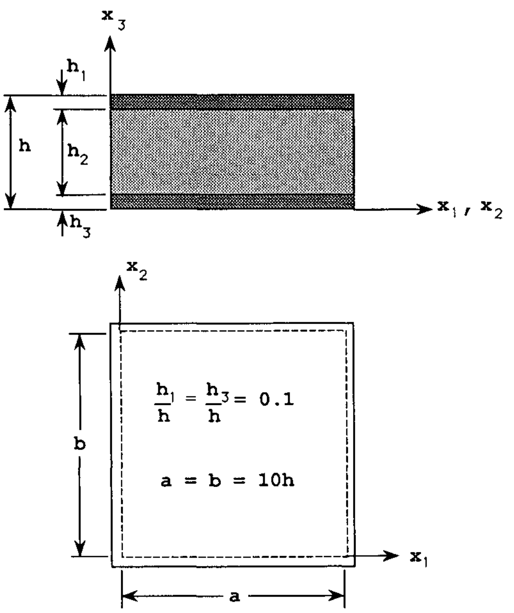


Fig. 4 Free-vibration test plate geometry.

completely defined by the kinematic and interlaminar stress field variables. Therefore, the number of independent field variables was reduced from $20N + 3$ to $10N + 3$ where N is the number of layers in the laminate,[$\bar{v}_\rho^{(k)}$, $\bar{\phi}_\rho^{(k)}$, $\bar{v}_3^{(k)}$, $\bar{\phi}_3^{(k)}$, $\bar{u}_3^{(k)}$, and $\bar{\sigma}_{i3}^{(k)}$]. The only governing functional specializations employed in the solution of the equations that limit the physical nature of the problem are the following:

1) Body force terms are assumed negligible.

2) No displacement or traction discontinuities at internal boundaries are present.

For the class of materials being considered, the contribution of the body force terms to the load vector is negligible. The second specialization limits the problem to one in which no interlaminar delaminations are present.

Numerical Examples

A finite element computer program incorporating the present theory and with an option to use either an eight-noded isoparametric element or a nine-noded heterosis element has been developed. To take advantage of the unique form of the mass and stiffness matrices, we used a condensation algorithm to reduce the number of degrees of freedom for the free vibration eigenanalysis. Since the $\bar{\sigma}_{i3}^{(k)}$ degrees of freedom are massless, they are statically condensed from the stiffness matrix. Similarly, $\bar{\phi}_3^{(k)}$ and $\bar{u}_3^{(k)}$ have zero stiffness contribution and are statically condensed from the mass matrix. This leaves $5N$ degrees of freedom per nodal point [$\bar{v}_\rho^{(k)}$, $\bar{\phi}_\rho^{(k)}$, and $\bar{v}_3^{(k)}$].

Since the elastodynamic laminated plate theory is a generalization of the static model developed by Pagano,[1] the formulation was initially checked by comparing the results obtained with those reported by Pagano for in-plane loading of cross-ply and angle-ply free-edge delamination coupons. Using the material properties and coupon geometries reported by Pagano[1], we calculated the distribution of stresses at the midplane of a four-ply $[0/90]_s$ laminate using the heterosis element. The finite element mesh discretized the laminate with 18 elements along its width and 2 elements along its length. Figure 3 presents σ_{23} plotted from the center of the coupon to the free edge for a coupon of width b. Note the present dynamic formulation degenerates to Pagano's solution for the static case.

Results of the present dynamic finite element formulation were compared with an exact solution for the free vibration of three-ply laminated plates as reported by Srinivas and Roa.[12] Each layer of the laminate was assumed to be orthotropic with the ratio of the elastic moduli given by

$$Q_{22}/Q_{11} = 0.543103 \qquad Q_{12}/Q_{11} = 0.233190$$

$$Q_{33}/Q_{11} = 0.530172 \qquad Q_{66}/Q_{11} = 0.262931$$

$$Q_{13}/Q_{11} = 0.010776 \qquad Q_{55}/Q_{11} = 0.159914$$

$$Q_{23}/Q_{11} = 0.098276 \qquad Q_{44}/Q_{11} = 0.266810$$

The geometry of the plate used in the test problem is shown in Fig. 4. The simply supported sandwich plate was composed of three layers with the top and bottom layers having the same thickness and material properties. Each of the three layers of the sandwich plate was modeled as a single layer.

Srinivas and Roa[12] presented the fundamental frequency from their exact solution for six different ratios of stiffness and density for the test plate shown in Fig. 4. The results are compared with the fundamental frequencies as computed by the present theory. The calculated fundamental natural frequencies for three levels of discretization for the square plate are shown in Table 1. Results of the theory are shown to be in excellent agreement for each of the six test cases.

In general, complementary formulations give upper bound solutions for displacement and lower bound solutions for natural frequencies. However, the present theory is a mixed formulation. Therefore, no general observations regarding the bounds of the solution can be made. As an example, for the last four test cases of Table 1, the calculated natural frequency is underpredicted when using a 3×3 mesh and slightly overpredicted when using a 4×4 mesh.

Numerical results for nondimensional frequencies of plates with thickness-width ratios of $h/a = 0.1, 0.2, \ldots, 0.5$ and $h/b = 0.1, 0.2, \ldots, 0.5$ are presented in Table 2. Each plate was modeled as one layer using 24–32 elements with the mesh geometry dependent on the plate aspect ratio. The results shown for the present theory are the frequencies corresponding to the lowest out-of-plane flexural mode of vibration. However, the reported frequencies for the present theory are not necessarily the fundamental frequencies of the plates. The fundamental mode and subsequent lower modes for several of the plates were in-plane modes; i.e., the lowest out-of-plane flexural mode was not the fundamental mode. The table identifies the plates with in-plane fundamental modes and reports the number of in-plane modes that have lower frequencies than the lowest out-of-plane flexural mode.

The convergence of the nondimensional fundamental frequency with mesh refinement for the first two test cases of Table 2 is shown in Figs. 5a and 5b. It is seen that the solution converges monotonically, and for the test cases shown, refinement beyond 24–32 elements yields minimal change in the results. Since the convergence is monotonic and not oscillatory, further refinement

Table 1 Nondimensional fundamental frequencies for sandwich plates

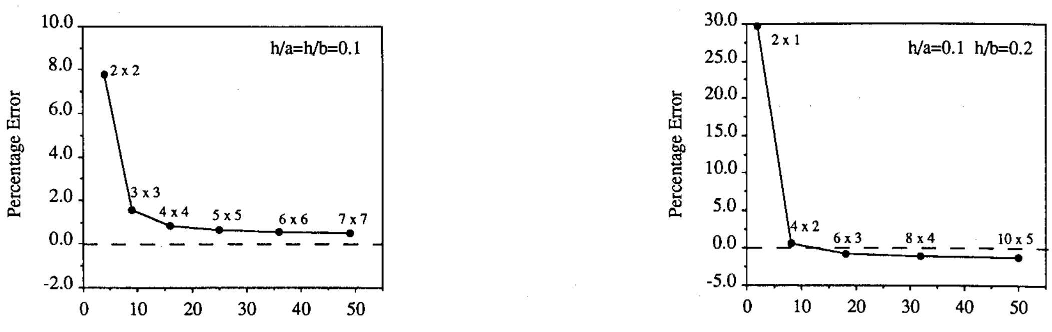
$\rho^{(1)}/\rho^{(2)}$	$Q_{11}^{(1)}/Q_{11}^{(2)}$	$\lambda = \omega h\sqrt{\rho^{(2)}/Q_{11}^{(2)}}$ 2 × 2 mesh	3 × 3 mesh	4 × 4 mesh	Exact, Ref. 12
1	1	0.049225 (+3.808%)[a]	0.047600 (+0.381%)	0.047602 (+0.386%)	0.047419
1	2	0.057702 (+1.158%)	0.057184 (+0.250%)	0.057184 (+0.250%)	0.057041
1	5	0.075666 (−1.921%)	0.076508 (−0.829%)	0.077304 (+0.202%)	0.077148
1	10	0.094621 (−3.550%)	0.096909 (−1.218%)	0.098375 (+0.276%)	0.098104
1	15	0.107367 (−4.165%)	0.110433 (−1.429%)	0.112412 (+0.337%)	0.112034
3	15	0.090688 (−4.081%)	0.093232 (−1.391%)	0.094886 (+0.358%)	0.094548

[a]Deviation from exact solution in Ref. 12.

Table 2 Lowest out-of-plane nondimensional frequencies for homogeneous plates

h/a	*h/b*	Present theory	Error, %	$\lambda = \omega h\sqrt{\rho/Q_{11}}$ Thin-plate theory[a]	Error, %	Exact solution[a]
0.1	0.1	0.04772	+0.63	0.04967	+4.74	0.04742
0.1	0.2	0.10219	−1.06	0.11200	+8.44	0.10329
0.1	0.3	0.18297[b]	−3.09	0.21537	+14.07	0.18881
0.1	0.4	0.28038[b]	−5.41	0.35993	+21.23	0.29690
0.1	0.5	0.38895[c]	−7.66	0.54574	+29.56	0.42124
0.2	0.1	0.12288	+3.43	0.13538	+13.95	0.11880
0.2	0.2	0.17203	+1.54	0.19866	+17.26	0.16942
0.2	0.3	0.24557	−0.79	0.30289	+22.37	0.24753
0.2	0.4	0.33626[b]	−3.25	0.44802	+28.91	0.34755
0.2	0.5	0.43830[b]	−5.59	0.63418	+36.59	0.46428
0.3	0.1	0.23323[b]	+6.96	0.27789	+27.45	0.21804
0.3	0.2	0.27525	+4.88	0.34176	+30.22	0.26244
0.3	0.3	0.33997	+2.40	0.44699	+34.64	0.33200
0.3	0.4	0.42147	−0.22	0.59313	+40.41	0.42242
0.3	0.5	0.51477[b]	−2.79	0.78011	+47.31	0.52956
0.4	0.1	0.36476[c]	+9.90	0.47734	+43.82	0.33189
0.4	0.2	0.40077[b]	+8.12	0.54152	+46.09	0.37066
0.4	0.3	0.45680	+5.67	0.64750	+49.80	0.43225
0.4	0.4	0.52856	+2.94	0.79466	+54.78	0.51342
0.4	0.5	0.61243	+0.25	0.98269	+60.85	0.61092
0.5	0.1	0.50980[d]	+12.62	0.72274	+62.10	0.45265
0.5	0.2	0.54026[b]	+10.98	0.79809	+63.95	0.48680
0.5	0.3	0.58844[b]	+8.64	0.90459	+67.02	0.54160
0.5	0.4	0.65133[b]	+5.96	1.0526	+71.25	0.61465
0.5	0.5	0.72588	+3.20	1.2416	+76.52	0.70338

[a]Exact solution and thin-plate values from Ref. 12.
[b]One lower in-plane mode calculated.
[c]Two lower in-plane modes calculated.
[d]Three lower in-plane modes calculated.



a) **Plate with aspect ratio 1:1** b) **Plate with aspect ratio 2:1**

Fig. 5 Solution convergence with mesh refinement.

of the mesh for $h/a = 0.1$ and $h/b = 0.2$ that is shown in Fig. 5b will result in a negligible increase in the error based on the exact solution given in Ref. 12.

Discussion

The present self-consistent theory imposes no displacement distribution assumptions on the field equations; rather, the governing assumption is a linear through-the-thickness in-plane stress distribution. Since the formulation is stress based, i.e., no displacement distribution assumptions are made, shear correction factors or shear coefficients are not required.

The governing field equations are derived by substituting the in-plane stress components into the equations of motion and solving for the out-of-plane shear and normal stress components. Using the six stress components, the plate equations of motion are derived. The formulation is generalized for a stacked plate or laminate analysis, and the resulting set of governing equations satisfies layer or lamina equilibrium, interlaminar stress continuity, and displacement continuity. The equations, which do not smear the properties of individual layers of the plate but represent each layer discretely, allow for nonzero laminate surface tractions.

The mixed formulation contains generalized stress and generalized displacement field variables as well as interlaminar transverse shear and normal stress field variables on a layer-wise basis. Because of the large number of field variables, the theory is computationally intense. However, the strength of the theory lies in the fact that the equations of motion are satisfied on a layerwise basis throughout the laminate, providing extremely accurate interlaminar stress predictions. In addition, accurate global dynamic behavior is predicted. Therefore, the use of the theory is advantageous when localized or interlaminar behavior is of interest.

Conclusions

Derivation of a self-consistent stress-based plate theory for modeling localized steady-state or transient dynamic behavior of layer media is presented. The theory differs from all existing dynamic plate theories since no displacement distribution assumption is made. The present theory does not require shear correction factors as do most plate theories.

A finite element formulation of the governing equations was derived, and results for the free-vibration analyses of simply supported sandwich plates are shown to be in excellent agreement with an exact solution. It has also been shown that the theory is capable of predicting the vibration behavior of thin as well as thick homogeneous, high-aspect-ratio plates with good accuracy.

Appendix

The operator matrices for the interlaminar stress and generalized displacement coupling are given by

$$[B]^{(k)} = \begin{bmatrix} -t^* & 0 \\ \frac{h_k}{2} t^* & 0 \\ 0 & -t^* \\ -\frac{3\rho}{5600} h_k^3 S_{3333}^{(k)} \frac{\partial}{\partial \gamma} & \frac{3\rho}{2800} h_k^2 S_{3333}^{(k)} \\ \frac{\rho}{240} h_k^3 S_{3333}^{(k)} \frac{\partial}{\partial \gamma} & 0 \\ -\frac{1}{12} h_k t^* S_{\mu\rho 33}^{(k)} \frac{\partial}{\partial \gamma} & \tfrac{1}{2} t^* S_{\mu\rho 33}^{(k)} \\ -\frac{1}{12} h_k t^* S_{3333}^{(k)} \frac{\partial}{\partial \gamma} & \tfrac{1}{2} t^* S_{3333}^{(k)} \\ \frac{1}{10} t^* S_{\mu\rho 33}^{(k)} \frac{\partial}{\partial \gamma} & -\frac{6}{5h_k} t^* S_{\mu\rho 33}^{(k)} \\ \frac{1}{10} t^* S_{3333}^{(k)} \frac{\partial}{\partial \gamma} & -\frac{6}{5h_k} t^* S_{3333}^{(k)} \\ -\frac{2}{5} t^* S_{\rho 3\gamma 3}^{(k)} & 0 \end{bmatrix} \qquad [C]^{(k)} = \begin{bmatrix} t^* & 0 \\ \frac{h_k}{2} t^* & 0 \\ 0 & t^* \\ -\frac{3\rho}{5600} h_k^3 S_{3333}^{(k)} \frac{\partial}{\partial \gamma} & -\frac{3\rho}{2800} h_k^2 S_{3333}^{(k)} \\ -\frac{\rho}{240} h_k^3 S_{3333}^{(k)} \frac{\partial}{\partial \gamma} & 0 \\ \frac{1}{12} h_k t^* S_{\mu\rho 33}^{(k)} \frac{\partial}{\partial \gamma} & \tfrac{1}{2} t^* S_{\mu\rho 33}^{(k)} \\ \frac{1}{12} h_k t^* S_{3333}^{(k)} \frac{\partial}{\partial \gamma} & \tfrac{1}{2} t^* S_{3333}^{(k)} \\ \frac{1}{10} t^* S_{\mu\rho 33}^{(k)} \frac{\partial}{\partial \gamma} & \frac{6}{5h_k} t^* S_{\mu\rho 33}^{(k)} \\ \frac{1}{10} t^* S_{3333}^{(k)} \frac{\partial}{\partial \gamma} & \frac{6}{5h_k} t^* S_{3333}^{(k)} \\ -\frac{2}{5} t^* S_{\rho 3\gamma 3}^{(k)} & 0 \end{bmatrix}$$

The operator for the generalized body force terms is

$$[D_u]^{(k)^T} = \begin{bmatrix} t^* & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & t^* & -\frac{3\rho}{2800} h_k^2 S_{3333}^{(k)} & 0 & 0 & 0 & \frac{1}{5h_k} t^* S_{\mu\rho 33}^{(k)} & \frac{1}{5h_k} t^* S_{\mu\rho 33}^{(k)} & 0 \end{bmatrix}$$

The body force vector is

$$\{F\}^{(k)^T} = \begin{bmatrix} F_\gamma^{(k)} & F_3^{(k)} \end{bmatrix}$$

The vector $\{E_u\}^{(k)^T}$ containing the in-plane boundary terms is given by

$$\{E_u\}^{(k)^T} = \left[0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ E_6^{(k)}\right]$$

and the initial conditions are given by

$$\{Z_u\}^{(k)^T} = \left[Z_1^{(k)}\ Z_2^{(k)}\ Z_3^{(k)}\ Z_7^{(k)}\ Z_8^{(k)}\ S_{\mu\rho33}^{(k)}\ Z_4^{(k)}\ S_{3333}^{(k)}Z_4^{(k)}\ S_{\mu\rho33}^{(k)}Z_5^{(k)}\ S_{3333}^{(k)}Z_5^{(k)}\ S_{\rho3\gamma3}^{(k)}Z_6^{(k)}\right]$$

The operator matrices for the interface continuity equations are given by

$$[B]^{(k)^T} = \begin{bmatrix} -t^* & 0 \\ \frac{h_k}{2}t^* & 0 \\ 0 & -t^* \\ \frac{3\rho}{5600}h_k^3S_{3333}^{(k)}\frac{\partial}{\partial\gamma} & \frac{3\rho}{2800}h_k^2S_{3333}^{(k)} \\ -\frac{\rho}{240}h_k^3S_{3333}^{(k)}\frac{\partial}{\partial\gamma} & 0 \\ \frac{1}{12}h_kt^*S_{\mu\rho33}^{(k)}\frac{\partial}{\partial\gamma} & \frac{1}{2}t^*S_{\mu\rho33}^{(k)} \\ \frac{1}{12}h_kt^*S_{3333}^{(k)}\frac{\partial}{\partial\gamma} & \frac{1}{2}t^*S_{3333}^{(k)} \\ -\frac{1}{10}t^*S_{\mu\rho33}^{(k)}\frac{\partial}{\partial\gamma} & -\frac{6}{5h_k}t^*S_{\mu\rho33}^{(k)} \\ -\frac{1}{10}t^*S_{3333}^{(k)}\frac{\partial}{\partial\gamma} & -\frac{6}{5h_k}t^*S_{3333}^{(k)} \\ -\frac{2}{5}t^*S_{\rho3\gamma3}^{(k)} & 0 \end{bmatrix}$$

$$[\bar{C}]^{(k)^T} = \begin{bmatrix} t^* & 0 \\ \frac{h_k}{2}t^* & 0 \\ 0 & t^* \\ \frac{3\rho}{5600}h_k^3S_{3333}^{(k)}\frac{\partial}{\partial\gamma} & -\frac{3\rho}{2800}h_k^2S_{3333}^{(k)} \\ \frac{\rho}{240}h_k^3S_{3333}^{(k)}\frac{\partial}{\partial\gamma} & 0 \\ -\frac{1}{12}h_kt^*S_{\mu\rho33}^{(k)}\frac{\partial}{\partial\gamma} & \frac{1}{2}t^*S_{\mu\rho33}^{(k)} \\ -\frac{1}{12}h_kt^*S_{3333}^{(k)}\frac{\partial}{\partial\gamma} & \frac{1}{2}t^*S_{3333}^{(k)} \\ -\frac{1}{10}t^*S_{\mu\rho33}^{(k)}\frac{\partial}{\partial\gamma} & \frac{6}{5h_k}t^*S_{\mu\rho33}^{(k)} \\ -\frac{1}{10}t^*S_{3333}^{(k)}\frac{\partial}{\partial\gamma} & \frac{6}{5h_k}t^*S_{3333}^{(k)} \\ -\frac{2}{5}t^*S_{\rho3\gamma3}^{(k)} & 0 \end{bmatrix}$$

and

$$[\Xi]^{(k)} = \begin{bmatrix} \Xi_{11}^{(k)} & \Xi_{12}^{(k)} \\ \Xi_{21}^{(k)} & \Xi_{22}^{(k)} \end{bmatrix}$$

where

$$\Xi_{11}^{(k)} = \frac{8}{15}t^*\left[h_kS_{\rho3\gamma3}^{(k)} + h_{k+1}S_{\rho3\gamma3}^{(k+1)}\right] - \left(\frac{1}{720} + \frac{1}{2800}\right)t^*\left[h_k^3S_{3333}^{(k)} + h_{k+1}^3S_{3333}^{(k+1)}\right]\frac{\partial^2}{\partial\gamma\partial\rho}$$

$$\Xi_{12}^{(k)} = \frac{1}{1400}t^*\left[h_k^2S_{3333}^{(k)} - h_{k+1}^2S_{3333}^{(k+1)}\right]\frac{\partial}{\partial\rho}$$

$$\Xi_{21}^{(k)} = -\frac{1}{1400}t^*\left[h_k^2S_{3333}^{(k)} - h_{k+1}^2S_{3333}^{(k+1)}\right]\frac{\partial}{\partial\rho}$$

$$\Xi_{22}^{(k)} = \frac{1}{700}t^*\left[h_kS_{3333}^{(k)} + h_{k+1}S_{3333}^{(k+1)}\right]$$

and

$$[\Lambda]^{(k)} = \begin{bmatrix} \Lambda_{11}^{(k)} & \Lambda_{12}^{(k)} \\ \Lambda_{21}^{(k)} & \Lambda_{22}^{(k)} \end{bmatrix}$$

where

$$\Lambda_{11}^{(k)} = -\frac{2}{15}t^*h_kS_{\rho3\gamma3}^{(k)} + \left(\frac{1}{720} - \frac{1}{2800}\right)t^*h_k^3S_{3333}^{(k)}\frac{\partial^2}{\partial\gamma\partial\rho}$$

$$\Lambda_{12}^{(k)} = -\frac{1}{1400}t^*h_k^2S_{3333}^{(k)}\frac{\partial}{\partial\rho}$$

$$\Lambda_{21}^{(k)} = -\frac{1}{1400}t^*h_k^2S_{3333}^{(k)}\frac{\partial}{\partial\gamma}$$

$$\Lambda_{22}^{(k)} = -\frac{1}{700}t^*h_kS_{3333}^{(k)}$$

also

$$[\bar{\Lambda}]^{(k)} = \begin{bmatrix} \bar{\Lambda}_{11}^{(k)} & \bar{\Lambda}_{12}^{(k)} \\ \bar{\Lambda}_{21}^{(k)} & \bar{\Lambda}_{22}^{(k)} \end{bmatrix}$$

where

$$\bar{\Lambda}_{11}^{(k)} = -\frac{2}{15}t^*h_kS_{\rho3\gamma3}^{(k)} + \left(\frac{1}{720} - \frac{1}{2800}\right)t^*h_k^3S_{3333}^{(k)}\frac{\partial^2}{\partial\gamma\partial\rho}$$

$$\bar{\Lambda}_{12}^{(k)} = \frac{1}{1400}t^*h_k^2S_{3333}^{(k)}\frac{\partial}{\partial\rho}$$

$$\bar{\Lambda}_{21}^{(k)} = \frac{1}{1400}t^*h_k^2S_{3333}^{(k)}\frac{\partial}{\partial\gamma}$$

$$\bar{\Lambda}_{22}^{(k)} = -\frac{1}{700}t^*h_kS_{3333}^{(k)}$$

and

$$[D_s]^{(k)} = \begin{Bmatrix} 0 & -\frac{3}{240}[h_k^2 S_{3333}^{(k)} + h_{k+1}^2 S_{3333}^{(k+1)}]\frac{\partial}{\partial\rho} \\ 0 & 0 \end{Bmatrix}$$

The vector containing the in-plane boundary terms is given by

$$\{E_s\}^{(k)^T} = \begin{bmatrix} E_a^{(k)} & 0 \end{bmatrix}$$

and the initial conditions are given by

$$\{Z_s\}^{(k)^T} = \begin{bmatrix} Z_a^{(k)} & Z_b^{(k)} \end{bmatrix}$$

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